Quadratic Pseudo-Boolean Optimization(QPBO): Theory and Applications At-a-Glance

Presented By:
Ahmad Al-Kabbany

Under the Supervision of:
Prof.Eric Dubois

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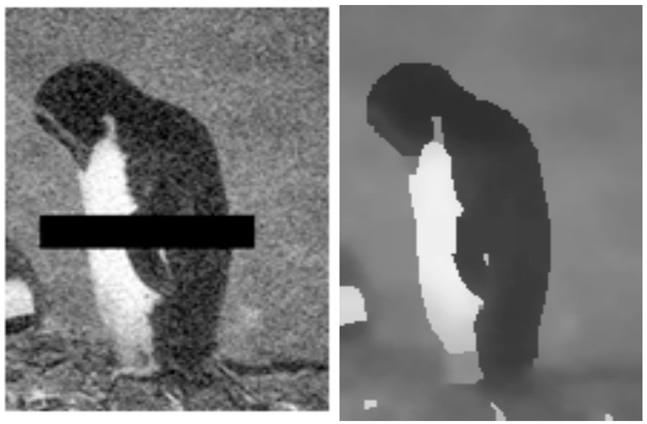
Outline

- Introduction
- The limitations of graph-cuts (GC)
- QPBO theory and properties
- What are we now able to do?
- Some results from the literature
- On-going work
- Key resources on QPBO

Where am I in the literature?

- Labelling problems in IP and CV
  - Intensities (restoration)
  - Disparities (stereo)
  - Motion vector components (optical flow)
  - Objects/surfaces (segmentation)

Where am I in the literature?



Where am I in the literature?

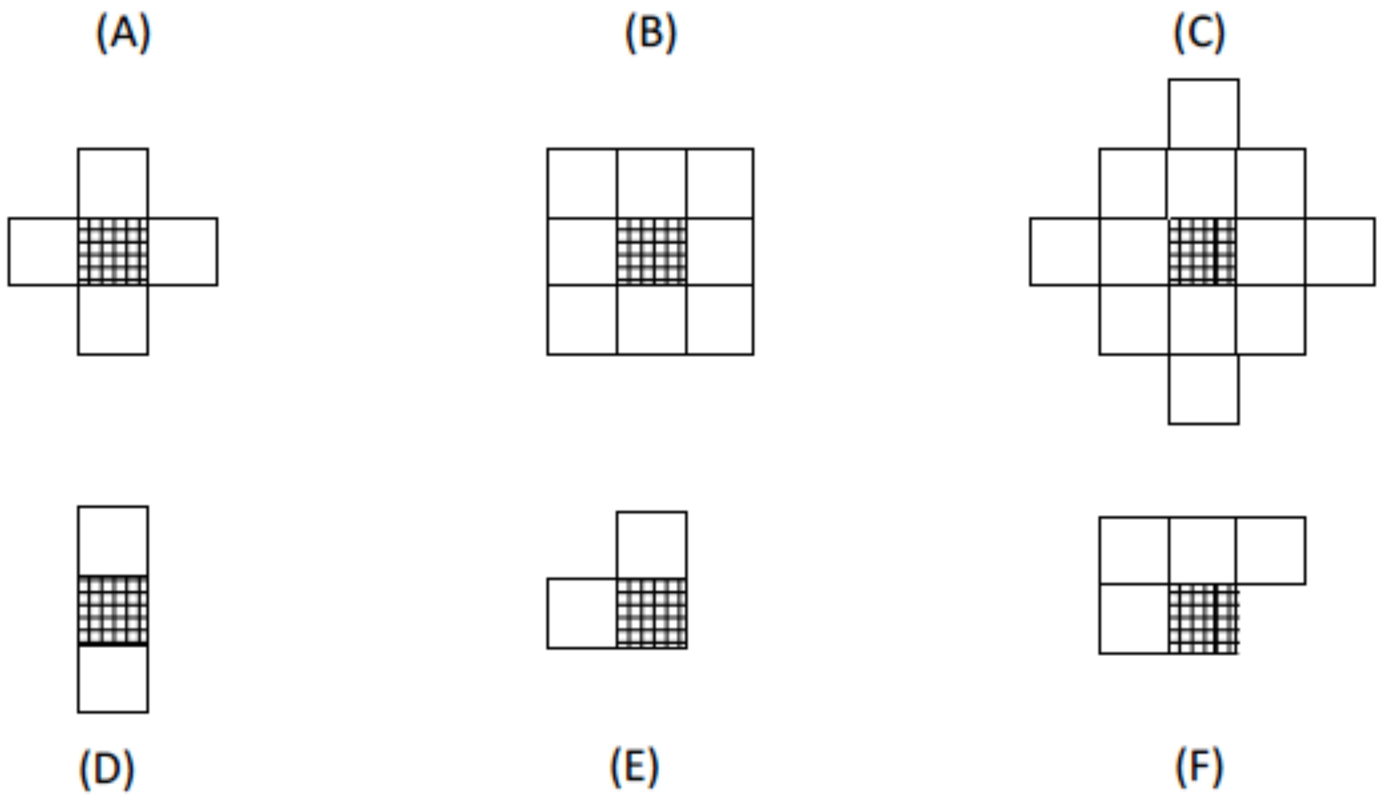
- Labelling problems in IP and CV
  - Intensities (restoration)
  - Disparities (stereo)
  - Motion vector components (optical flow)
  - Objects/surfaces (segmentation)
- Formulated as optimization problems
  - Minimizing a certain energy function
  - Approximate inference on graphical models
    - Markov random fields

Where am I in the literature?

The Markovian Property → $P(x_p / x_{S-p}) = P(x_p / x_{N-p})$

- Gibbs-Markov random field model
- Neighbourhood system
  - Defined on a sampling structure *Γ*, a subset of *Λ*
  - Given a site "x" in *Γ*, and N_x a subset of *Γ*
  - $\{N_x, x\}$ is a valid neighbourhood system if:
    - "x" doesn't belong to its neighbourhood
    - If "y" is in N_x then "x" is in N_y

Where am I in the literature?



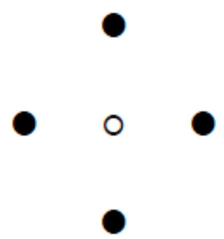
Prof.Eric Duboi's notes of ELG5378-Image Processing and Communications

Where am I in the literature?

- A clique “c” is a subset of *Γ*, such that either:
  - “c” is a single site in Γ, or
  - For all pairs of points (x,z) ∈ “c”, x ∈ N_z

First order neighbourhood system

Neighborhood



Cliques: single pixel ●; horizontal neighbors ● ●; vertical neighbors ● ●

Prof.Eric Duboi's notes of ELG5378-Image Processing and Communications

Where am I in the literature?

$$E(x)=\sum_{p\in V}\theta_p(x_p)+\sum_{(p,q)\in V}\theta_{pq}(x_p,x_q)$$

Unary term (observed data)

Pairwise Interaction term/potential (prior knowledge)

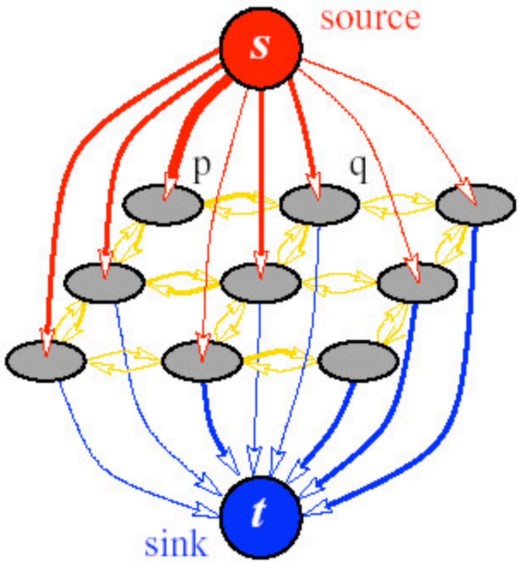
Where am I in the literature?

- Graph cuts in IP and CV
  - A type of combinatorial optimization
  - Discrete label space
  - Graph theoretical concepts
    - Flow network with set of vertices *V* and set of arcs *A*, each has a capacity
    - Two terminal nodes; a source and a sink
  - Vertices are pixels, terminal nodes are labels
  - A cut on a graph and the cut cost

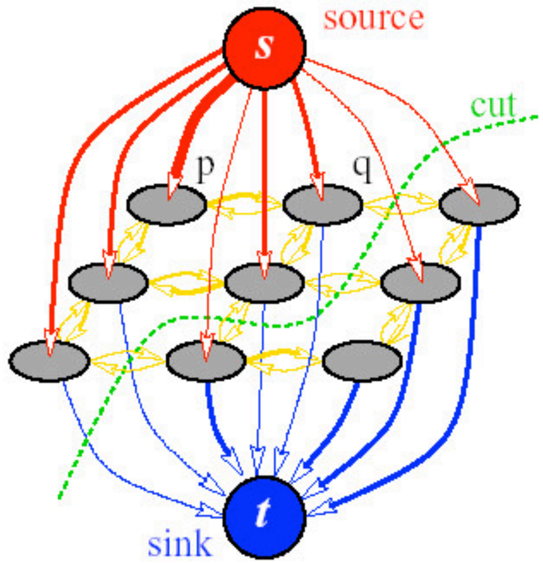
Where am I in the literature?

- The min cut
- The max flow/min cut theorem
- Max flow/min cut optimization (to minimize E)
- The the main idea
- The roadmap
  - Construct the graph
  - Compute the max flow and the min cut
  - Assign labels based on the min cut

Optimization with GC



(a) A graph $\mathcal{G}$



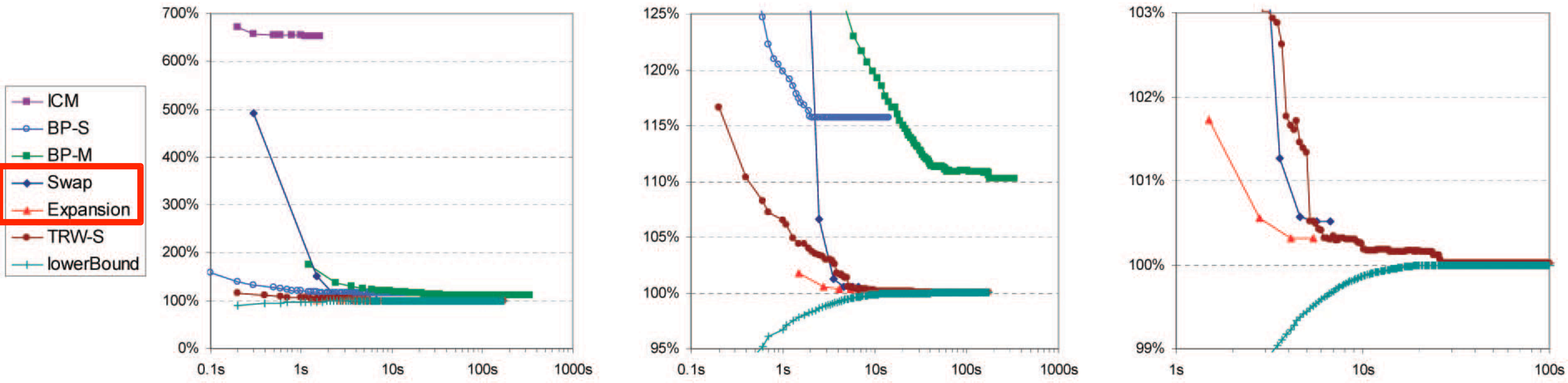
(b) A cut on $\mathcal{G}$

$$E(x) = \sum_{p \in V} \theta_p(x_p) + \sum_{(p,q) \in V} \theta_{pq}(x_p, x_q)$$

Optimization with GC

- Binary problems, exactly solvable, but…
- Multi-label problems
  - Near global optimum solutions
  - Move making algorithms (“Bianry-zing” the problem)
    - Approximate energy minimization iterative algorithms
    - α-expansion move (retain the label or change to α)
    - α-ß swap move (work with pairs of labels, swap or not based on what it now has)

Optimization with GC



A Comparative Study of Energy Minimization Methods for Markov Random Fields with Smoothness-Based Priors
R.Szeleiski, R.Zabih, D.Scharstein, O.Veksler, V.Kolmogorov, A.Agarwala, M.Tappen and C.Rother

What is the problem with GC ?

- What energy functions can be minimized?
- Binary and Multi-label MRF optimization
- The graph depends on the exact form of the potentials and the label space [1]

What is the problem with GC ?

$$E(x)=\sum_{p\in V}\theta_p(x_p)+\sum_{(p,q)\in V}\theta_{pq}(x_p,x_q)$$

Theorem 4.1 ($\mathcal{F}^2$ Theorem). *Let E be a function of n binary variables from the class $\mathcal{F}^2$, i.e.,*

$$E(x_1,\ldots,x_n)=\sum_i E^i(x_i)+\sum_{i<j}E^{i,j}(x_i,x_j). \quad (6)$$

Then, E is graph-representable if and only if each term $E^{i,j}$ satisfies the inequality

$$E^{i,j}(0,0)+E^{i,j}(1,1)\leq E^{i,j}(0,1)+E^{i,j}(1,0). \quad (7)$$

What is the problem with GC ?

- Sub-modularity

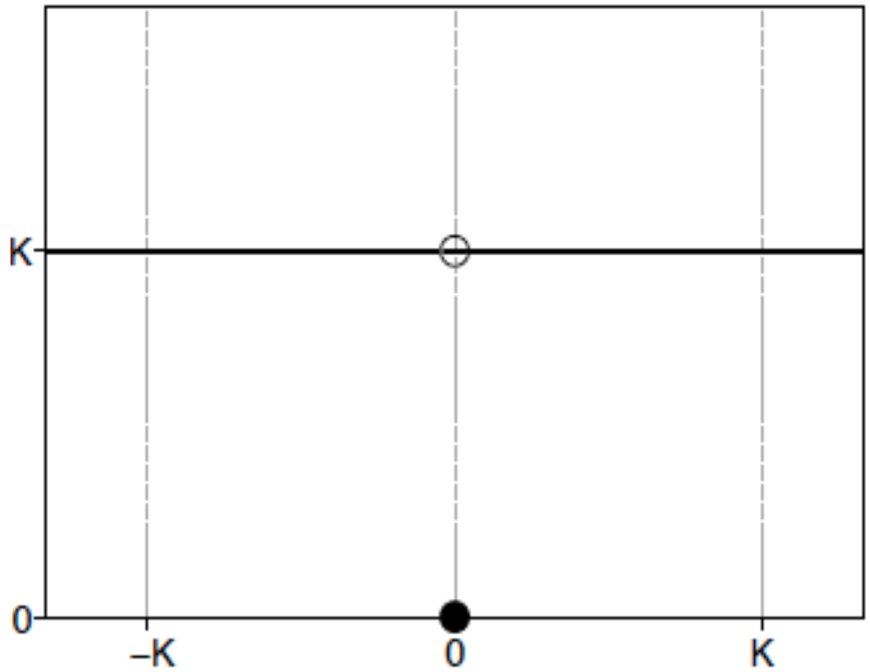
$$\theta_{pq}(0,0) + \theta_{pq}(1,1) \leq \theta_{pq}(0,1) + \theta_{pq}(1,0)$$

- The multi-label case

$$\min_{\mathbf{y}} E(\mathbf{x}) \text{ where } x_m = (1 - y_m)x_m + y_m\alpha$$

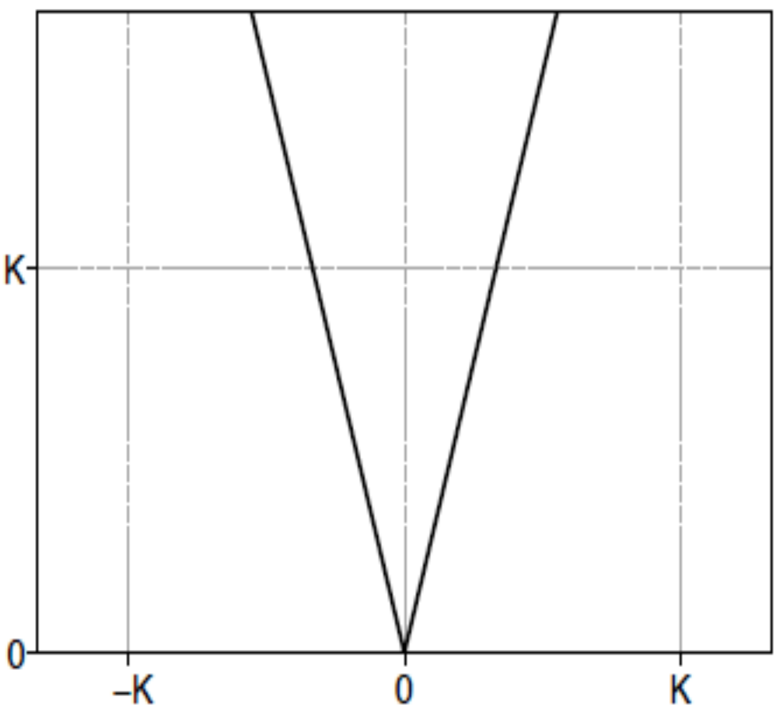
- Truncated GC (image stitching)

What is the problem with GC ?

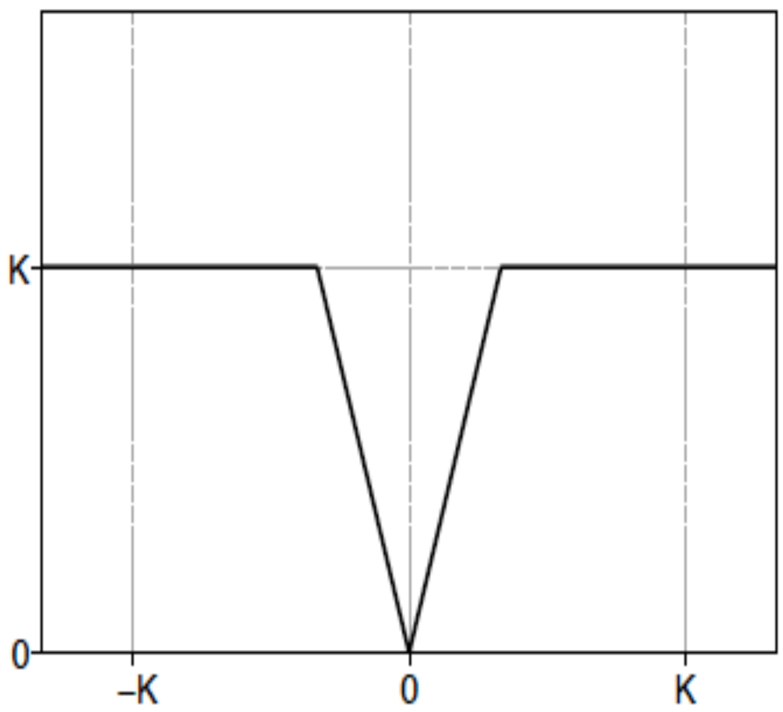


$$V_K^{\mathbf{Potts}}(\alpha, \beta) = K \cdot T(\alpha \neq \beta)$$

What is the problem with GC ?

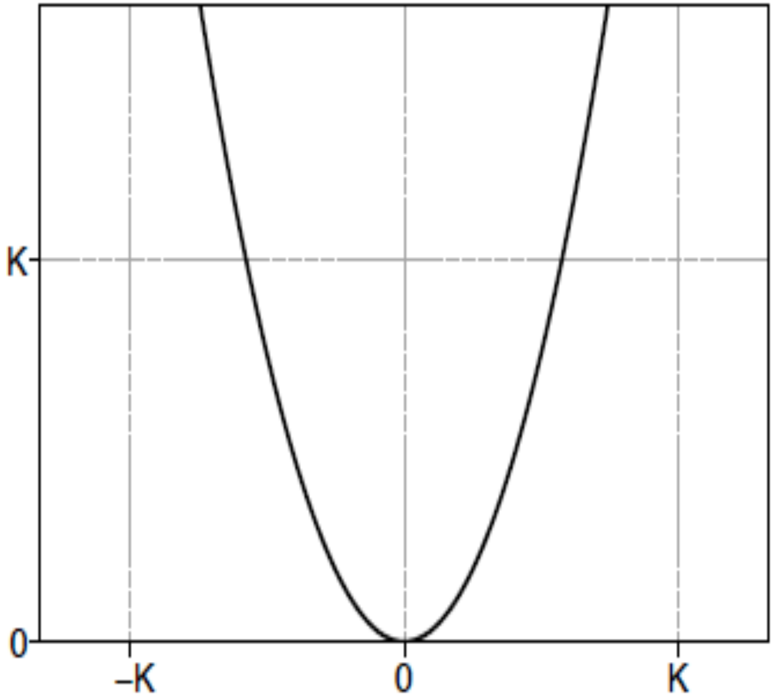


$$V^{\text{linear}}(\alpha,\beta) = |\alpha - \beta|$$

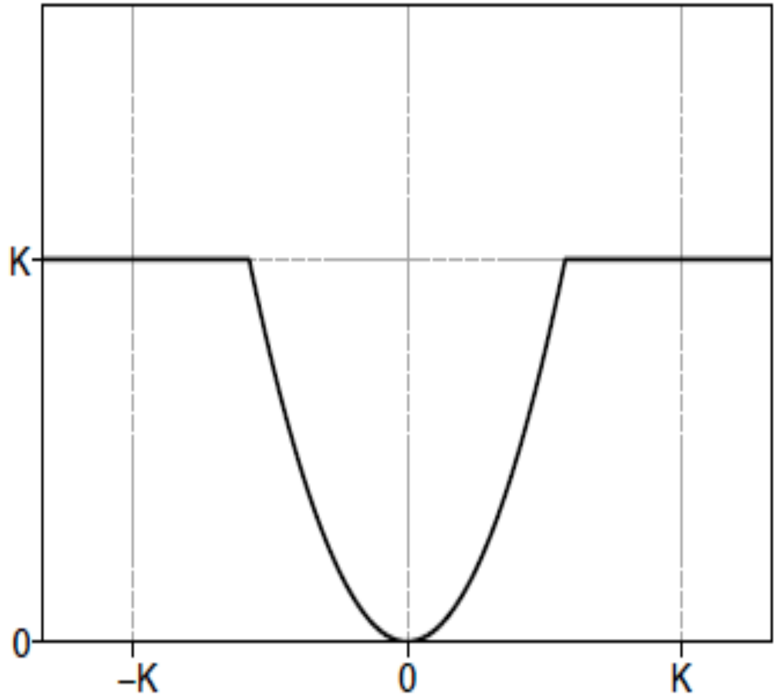


$$V_K^{\text{linear}}(\alpha,\beta) = \min\{\,|\alpha - \beta|, K\,\}$$

What is the problem with GC ?

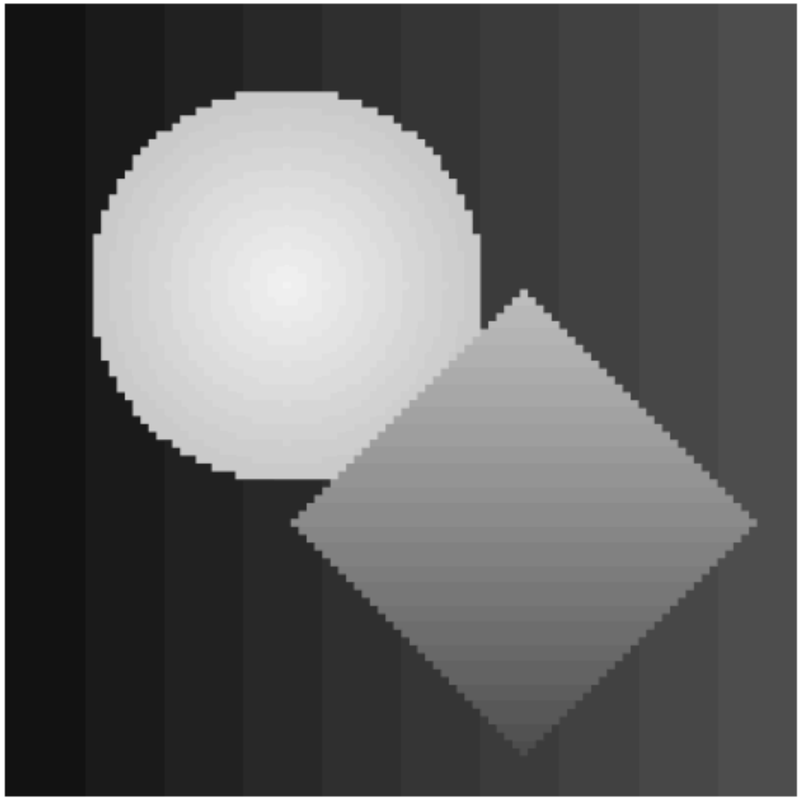


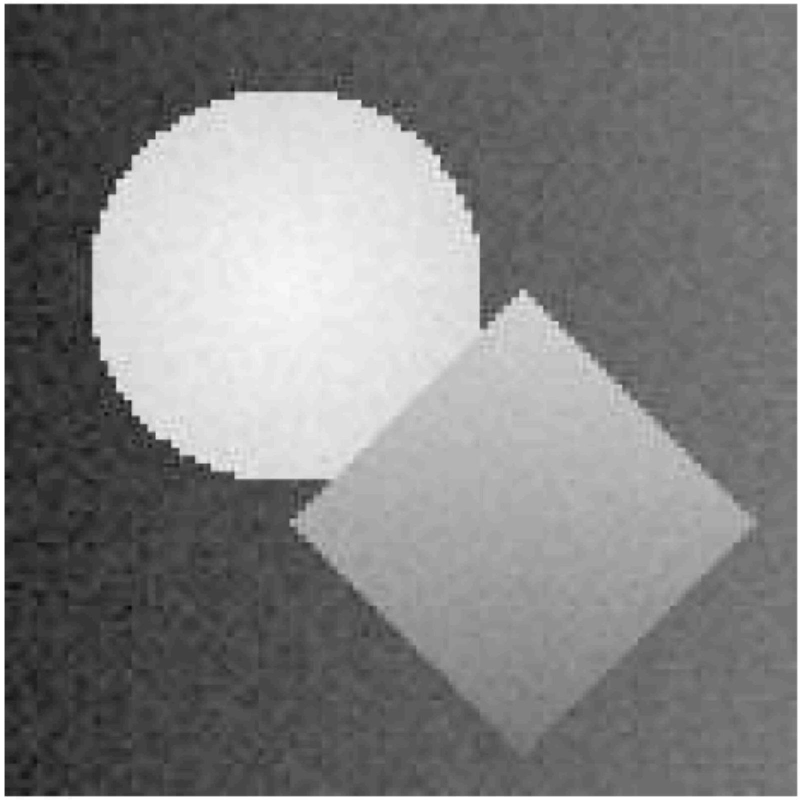
$$V^{\text{quadratic}}(\alpha, \beta) = (\alpha - \beta)^2$$



$$V_K^{\text{quadratic}}(\alpha, \beta) = \min\{ (\alpha - \beta)^2, K \}$$

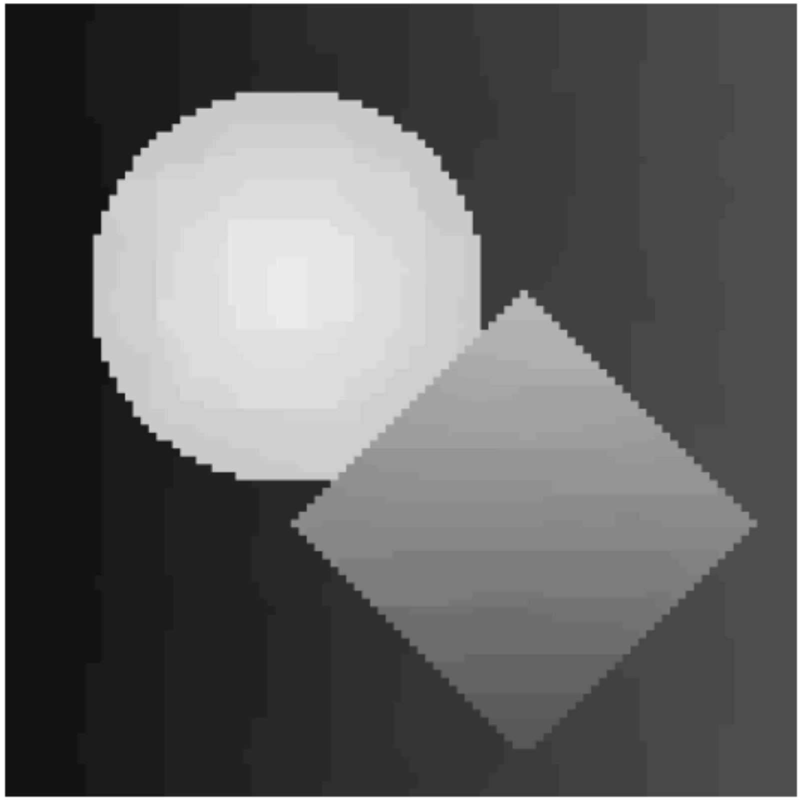
What is the problem with GC ?



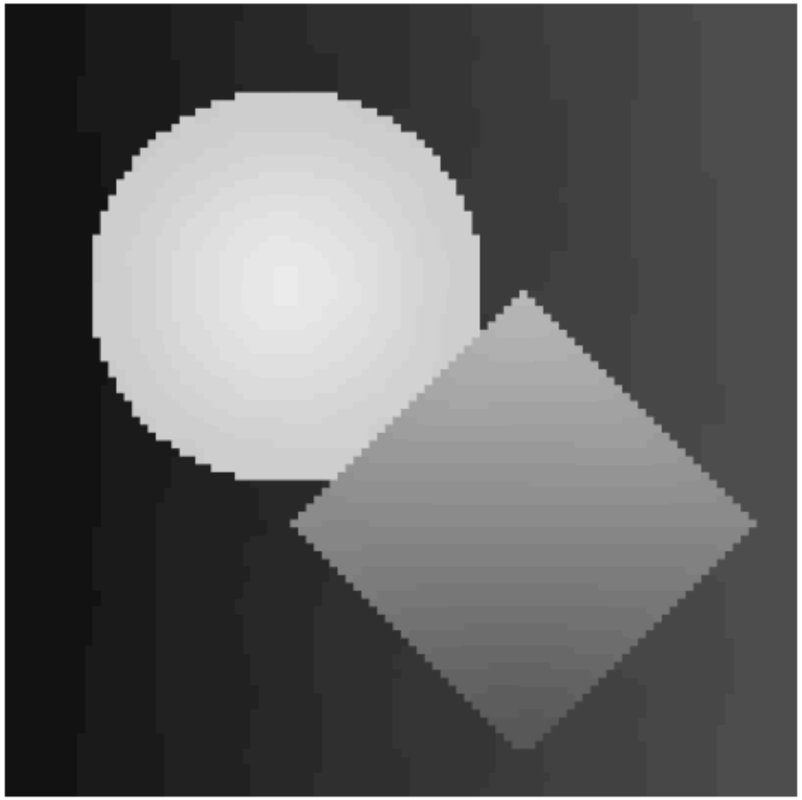


Graph cut-based optimization for MRF's with truncated convex priors
Olag Vekseler,CVPR, 2007

What is the problem with GC ?



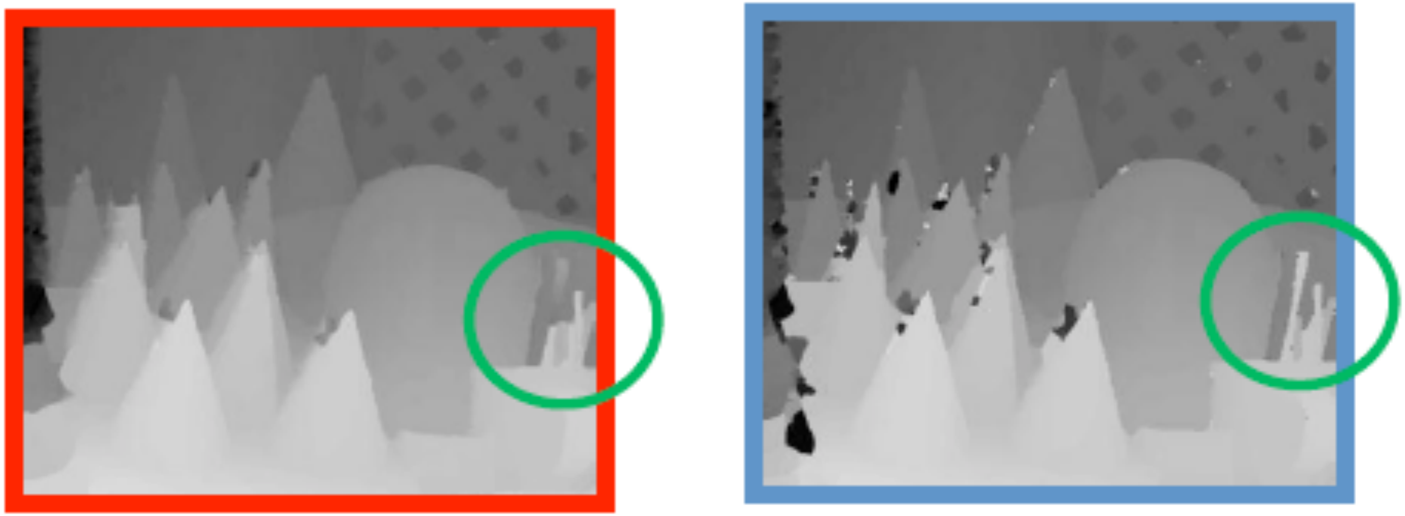
α-expansion move



range move

Graph cut-based optimization for MRF's with truncated convex priors
Olag Vekseler,CVPR, 2007

What is the problem with GC ?



No truncation (global min.)

with truncation (NP hard optimization)

DiscreteModels:Optimization and Appliactions
Carsten Rother, Microsoft Summer School of Computer Vision, Russia, 2011

What is the problem with GC ?

- The priors that suit the application the most
- What we are modelling
  - Higher-order clique potentials
- The framework of the optimization problem (learning)

QPBO Theory

Why Pseudo-Boolean?

- Boolean domain and real range; i.e. $f : \mathbb{B}^n \mapsto \mathbb{R}$
- The multi-linear polynomial form

$$f(x)=\sum_{i} a_i x_i+\sum_{i<j} a_{ij} x_i x_j+\sum_{i<j<k} a_{ijk} x_i x_j x_k+\ldots$$

Why Quadratic?

- Degree of f is the degree of the polynomial

$$E(x)=\sum_{p\in V} \theta_p(x_p)+\sum_{(p,q)\in V} \theta_{pq}(x_p,x_q)$$

QPBO Theory

- Example:

$$f : \mathbb{B}^n \mapsto \mathbb{R}$$

$$g(x_1, x_2, x_3, x_4) = 5x_1 + 13x_3 - 4x_1x_3 - 4x_2x_3 - 9x_3x_4 + 4x_1x_2x_3 + 7x_1x_2x_4$$

$\mathbf{x}$				$g(\mathbf{x})$
0	0	0	0	0
0	0	0	1	0
0	0	1	0	13
0	0	1	1	4
0	1	0	0	0
0	1	0	1	0
0	1	1	0	9
0	1	1	1	0
1	0	0	0	5
1	0	0	1	5
1	0	1	0	14
1	0	1	1	5
1	1	0	0	5
1	1	0	1	12
1	1	1	0	14
1	1	1	1	12

QPBO Properties

- Function degree reduction
  - Reduction to quadratic PBO

The optimization of a pseudo-Boolean function can always be reduced in polynomial time to the optimization of a quadratic pseudo-Boolean function.

Rosenberg, I.G. Reduction of bivalent maximization to the quadratic case. Cahiers du Centre d'Etudes de Recherche Operationnelle 17 (1975), 71-74

$$f(x_1, x_2, x_3, x_4, x_5) \stackrel{\text{def}}{=} 5x_1x_2 - 7x_1x_2x_3x_4 + 2x_1x_2x_3x_5$$

$$f^7 = 45x_6 + 45x_7 + 20x_1x_2 - 30x_1x_6 - 30x_2x_6 + 15x_3x_6 - 30x_3x_7 - 7x_4x_7 + 2x_5x_7 - 30x_6x_7.$$

  - Higher-order clique reduction to pair-wise potentials

QPBO Theory

- What is a roof dual?
  - Given f is a quadratic polynomial
  - Roof dual vs. floor dual
  - The QPBO method output
- Properties
  - Persistency (Weak Autarky)
  - Partial optimality (Weak Persistency)

QPBO Theory

- Weak Autarky
  - If y is a complete labelling while x is not
  - And if z=FUSE(x,y)
  - $E(z) \leq E(y)$
- Partial Optimality
  - There exists a global minimum labelling x^* such that $x=x^*$ for all labelled pixels p in x

QPBO Shortcomings

- Partial labelling
  - Only if non-sub-modularity exists
  - Application dependent
  - Usually bearable

NVS example

Ground Truth | Ground Truth (zoom) | QPBO (0.7s) | Graph Cut (0.3s)

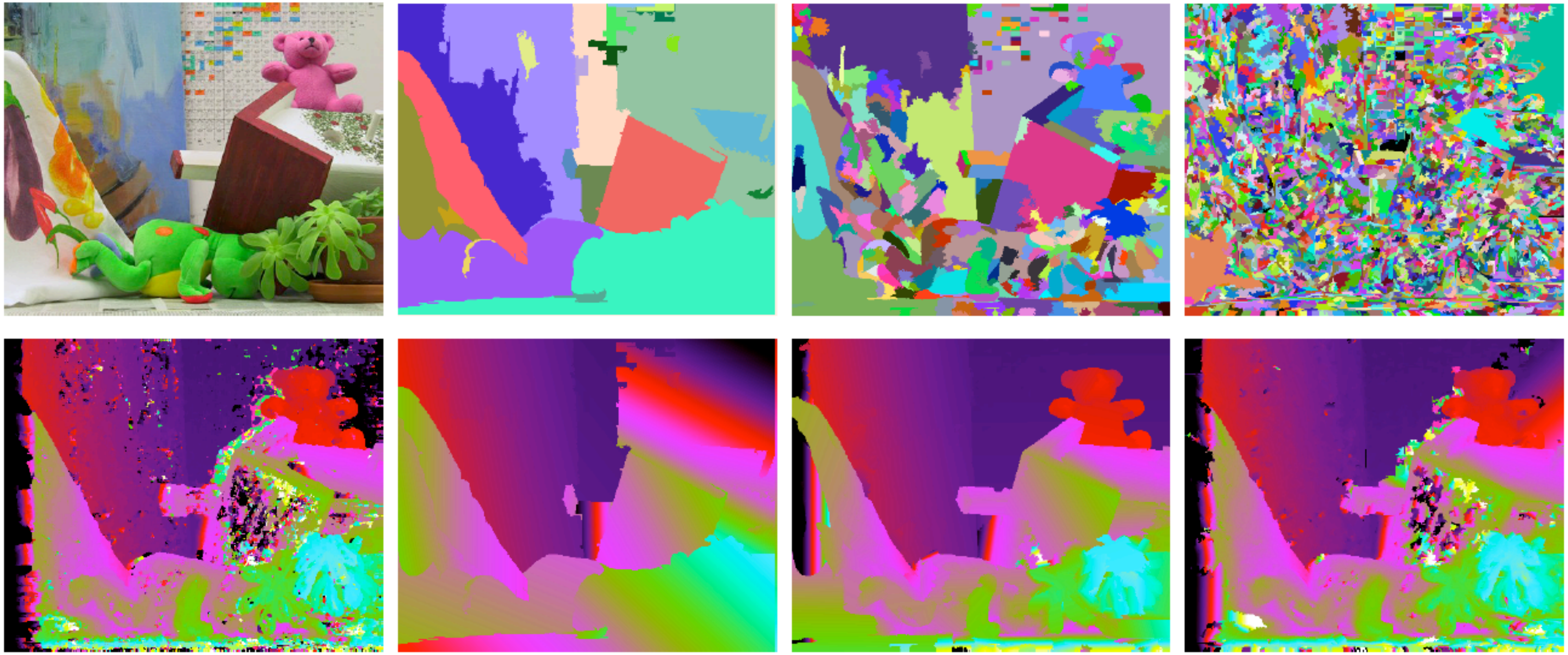
What are we now able to do?

- Higher-order clique potentials
  - Modelling rich statistics of natural scenes
  - Graph constructions were known
    - Potentials on cliques of size “n”; e.g. P^n Potts model
    - ex. triple cliques (2^{nd} order smoothness priors)
  - Graphs, in general, are non-sub-modular though
  - Global optimum finders were missing

What are we now able to do?

- The **fusion move** (generalized α-expansion)
  - "Binary-zing" the problem
  - Fusing 2 solutions (proposals) at a time

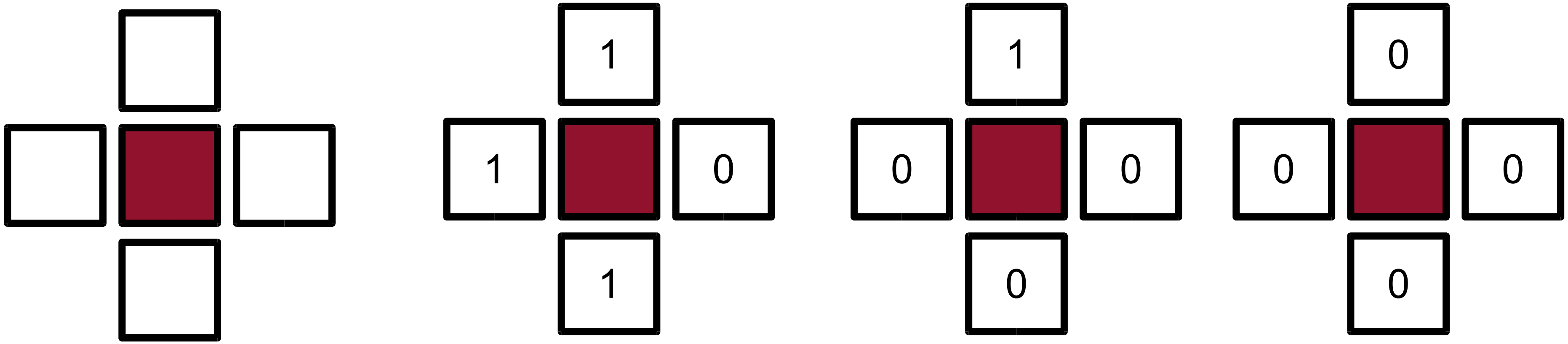
What are we now able to do?



SegPln Proposal Generation

Global Stereo Reconstruction under Second rder Smoothness Priors
O.J.Woodford, P.H.S.Torr, I.D.Reid, A.W.Fitzgibbon

What are we now able to do?



4-neighbourhood system

$$\theta_{pq}(0,0) + \theta_{pq}(1,1) \leq \theta_{pq}(0,1) + \theta_{pq}(1,0)$$

Fusing two solutions is generally non-sub-modular, NO MATTER WHAT PRIORS WE USE !

QPBO Flavours (Fusion Strategies)

As summarized in [8]

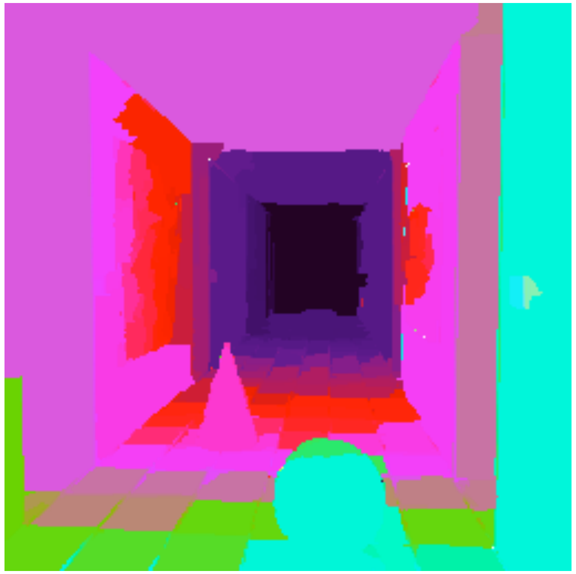
QPBO-F	***Fix to current***: fix unlabelled nodes to 0
QPBO-L	***Lowest energy label***: fix unlabelled nodes collectively to whichever of 0 or 1 that gives the lowest energy.
QPBOI-F	***Fix to current and improve***: fix unlabelled nodes to 0, and transform this labelling using QPBOI.
QPBOP	***Probe***: probe the graph to find the labels for more nodes, as a part of an optimal solution.
QPBO-R	***Lowest cost label per region***: split the unlabelled nodes to strictly connected regions and label each SCR(all affiliated cliques) with 0 or 1.

(% of unlabelled pixels vs. Time per fusion) OR (Number of fusions vs. Energy decrease per fusion)

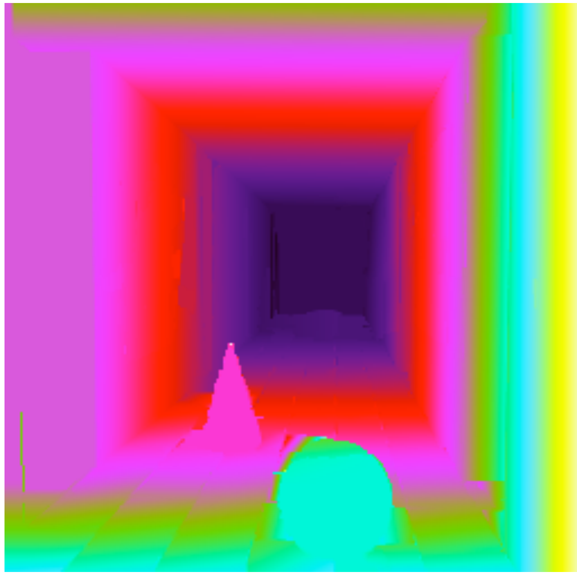
Some results from the literature

- Inference with triple cliques

Reference Image



Disparity map(1st order prior)

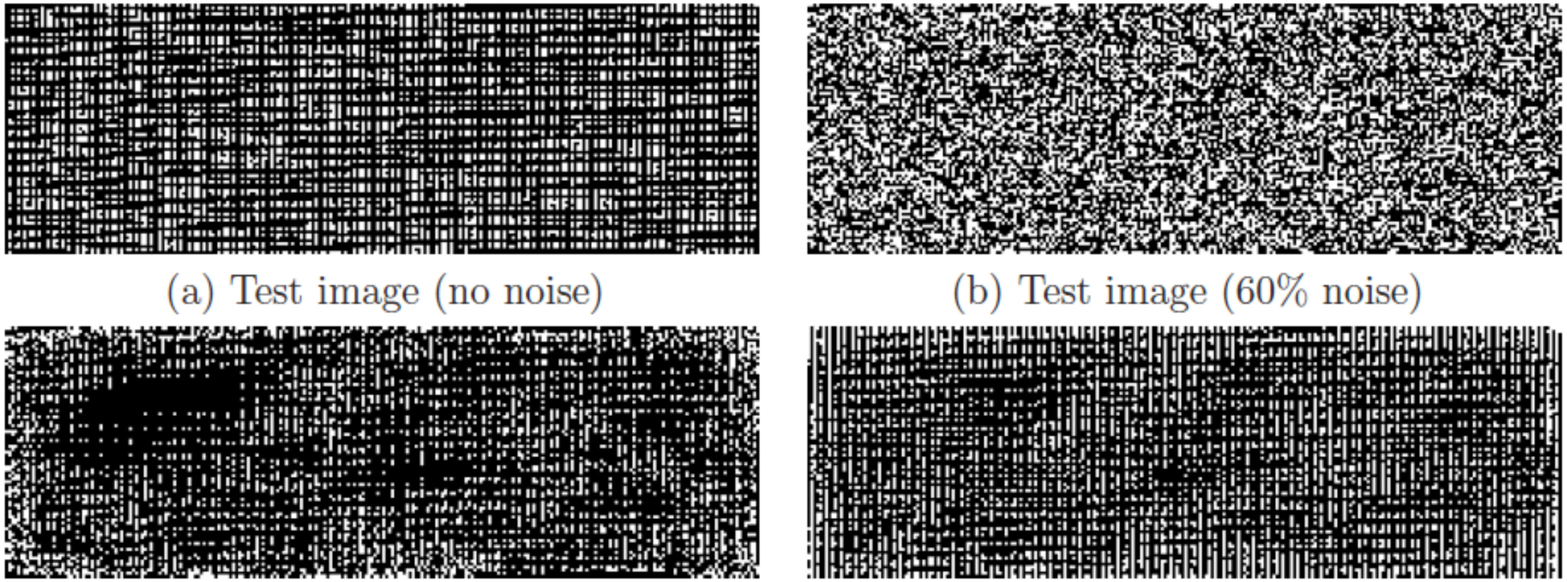


Disparity map(2nd order prior)

Global Stereo Reconstruction under Second rder Smoothness Priors
O.J.Woodford, P.H.S.Torr, I.D.Reid, A.W.Fitzgibbon

Some results from the literature

- Binary texture restoration



(a) Test image (no noise)

(b) Test image (60% noise)

(c) Submodular graph cuts (error 18.1)

(d) QPBO (error 16.0)

Minimizing non-sub-modular functions with graph cuts-a review
V.Kolmogorov and C.Rother

Some results from the literature

- Parallelized α-expansion

Solution1; E=2046

Solution2; E=2915

Global Stereo Reconstruction under Second rder Smoothness Priors
O.J.Woodford, P.H.S.Torr, I.D.Reid, A.W.Fitzgibbon

Some results from the literature

- Parallelized α-expansion

Fusion; E=1362

Alpha Expansion; E=1365

Global Stereo Reconstruction under Second rder Smoothness Priors
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Some results from the literature

- Object Stereo
  - Fusing colour, depth and 3D connectivity information

Object Stereo-Joint Stereo Matching and Object Segmentation
M.Bleyer, C.Rother, P.Kohli, D.Scharstein and S.Sinha

Some results from the literature

- Object Stereo
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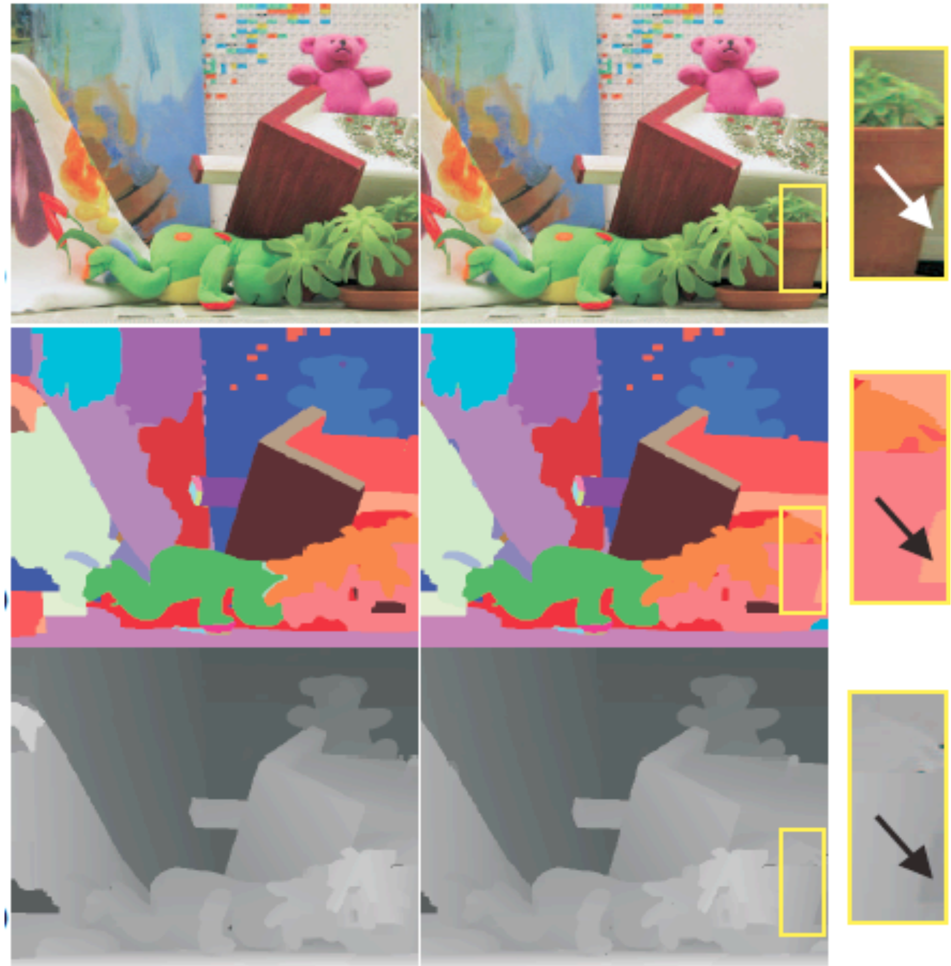

Object Stereo-Joint Stereo Matching and Object Segmentation
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Conclusions

- An optimization technique:
  - Capable to deal with non-sub-modularity
  - Capable to simplify general higher-order cliques
  - Capable to fuse solutions
  - Computationally complex
  - May produce incomplete solution

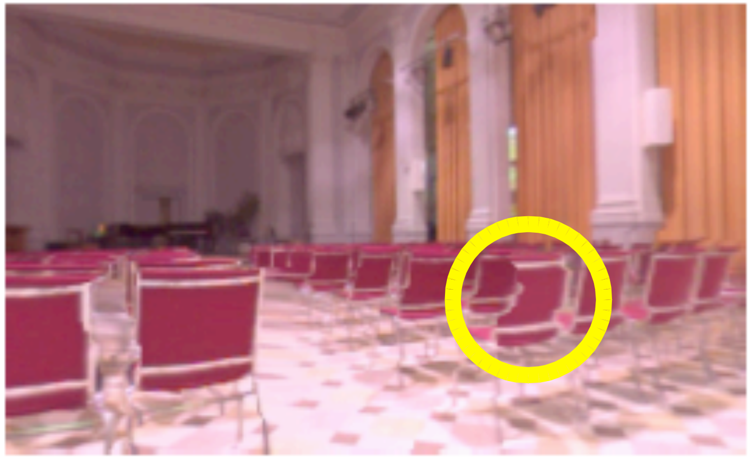
On going work

- Ill-posed problems of new view synthesis
  - Occlusions
    - Proposal management
  - Illumination modelling and shadow detection
  - Object-based NVS
- Real-time QPBO

On going work

Experiments - Preliminary Results

Novel view synthesis - Graph Cut



Presentation on Spherical Stereo
Alan Brunton et Al., VIVA Lab, 15 July 2011

Resources on QPBO(Publications)

1 V.Kolmogorov and R.Zabih, What Enrgy Functions Can Be Minimized via Graph Cuts?

2 Boros; Hammer (2002). "Pseudo-Boolean Optimization". Discrete Applied Mathematics

3 Hammer, P.L., P. Hansen and B. Simeone. Roof duality, complementation and persistency in quadratic 0-1 optimization. Mathematical Programming 28 (1984), pp. 121-155.

4 E. Boros, P. L. Hammer, and X. Sun. Network flows and minimization of quadratic pseudo-Boolean functions. Technical Report RRR 17-1991, RUTCOR, May 1991.

5 E. Boros, P. L. Hammer, and G. Tavares. Local search heuristics for unconstrained quadratic binary optimization. Technical Report RRR 9-2005, RUTCOR, Feb. 2005.

6 E. Boros, P. L. Hammer, and G. Tavares. Preprocessing of unconstrained quadratic binary optimization. Technical Report RRR 10-2006, RUTCOR, Apr. 2006.

7 V. Kolmogorov and C. Rother. Minimizing non-submodular functions with graph cuts - a review. PAMI, 29(7):1274-1279, 2007.

8 C. Rother, V. Kolmogorov, V. Lempitsky, and M. Szummer. Optimizing binary MRFs via extended roof duality. Technical Report MSR-TR-2007-46, Microsoft Research, 2007

9 P. Kohli, M. Kumar, and P. Torr. P3 & beyond: Solving energies with higher order cliques. In CVPR, 2007

10 V. Lempitsky, C. Rother, and A. Blake. Logcut - efficient graph cut optimization for Markov Random Fields. In ICCV, 2007.

11 O.J.Woodford, P.H.S.Torr, I.D.Reid, A.W.Fitzgibbon. Global Stereo Reconstruction under second Order Smoothness Priors, CVPR, 2008

Resources on QPBO(Software)

- **Vladimir Kolmogorov's** C++ implementation of different QPBO flavours:

  http://pub.ist.ac.at/~vnk/software.html

- **Oliver Woodford's** Matlab implementation of two publications of his that employed QPBO:

  http://www.robots.ox.ac.uk/~ojw/software.htm

- For related advancements:

  http://research.microsoft.com/en-us/projects/discoptimcomputervision/default.aspx

Thank you